The History of the Cycloid Curve

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Abstract

During the 17th century prominent mathematicians became fascinated with the cycloid curve. It was their favorite example to use in the development of their new ideas and theorems. They would use it to help them in their new discoveries because the unique properties aligned with the curve. Since it was used by most significant mathematicians of the 17th century, it is important to examine the history of it during this time period. This paper aims to examine mathematicians work with the curve, find similar methods used among mathematicians, and find how the cycloid curve aided the development of calculus.

In order to answer this question, proofs relating to finding general methods of tangents, areas, arc length to the curve and the brachistochrone problem were examined and analyzed. The methods used by mathematicians will be explained, compared, and contrasted. The relationship to the development of calculus will also be considered and explained. The goal of this paper is to illustrate the importance of the curve and why it must be studied will be demonstrated in this paper.

Contents

1 Introduction

Mathematics during the 17th century was a period of heightened creativity and innovation, developing into one of the greatest centuries in the history of mathematics. New ideas and techniques were being developed rapidly that expanded many fields of mathematics and even created new ones. One major topic of interest was the study of the cycloid curve. This is the curve that is generated by a point on the circumference of a circle as it rolls along a straight line.

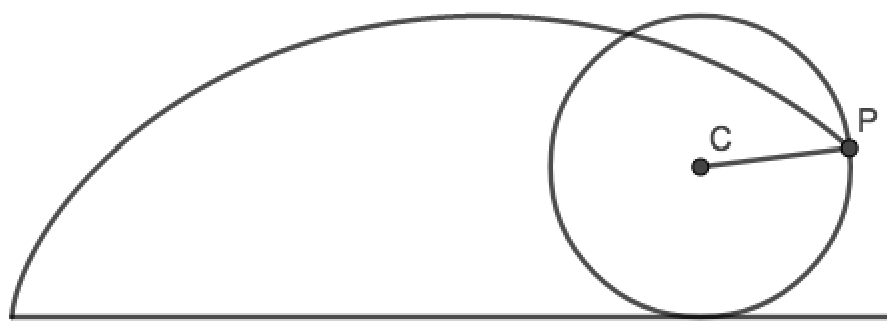

Figure 1: Cycloid

While the cycloid was discovered centuries earlier, it wasn't until the 17th century that it became a new and powerful tool for the study of curves. The cycloid was utilized to find new methods of tangents to curves, areas under curves, and much more. While its study contributed to many aspects of mathematics, an important one to focus on is its contribution to calculus. Calculus at this time was in the beginning stages, and the study of the cycloid helped aid mathematicians in their development of the subject. With such unique properties, the cycloid contributed to the discovery of new ideas in mathematics during the 17th century. This paper examines relationships among general methods for tangents, areas, and lengths of the curve and solutions to the Brachistochrone problem along with connections to the development of calculus.

2 Tangents

One considerable development aided by the cycloid curve was the study of tangents. 17th century mathematics involved an increasing interest in finding the general method of tangents because of recent fascination with dynamics and intellectual interests in the study of curves [5]. This heightened interest brought most, if not all, mathematicians of the time period to be involved with finding the general method of tangents. The cycloid curve is crucial in this development because the unique properties provided substantial evidence with the additional case to support each mathematician's methodology. The two leading and successful pioneers of this field of tangents during the 17th century were Gilles de Roberval and Pierre de Fermat.

2.1 Roberval

Gilles de Roberval (1602-1675) produced his first, now discarded, general method of tangents in 1636 and released his second method that proved to be fundamental in the modern theory of tangents by the early autumn of 1638. His mechanical method consisted of composition of motions, being that he considered every curve as a path of a moving point.

Theorem 2.1: The net force of a point E on the cycloid is the tangent to the curve at that point E.

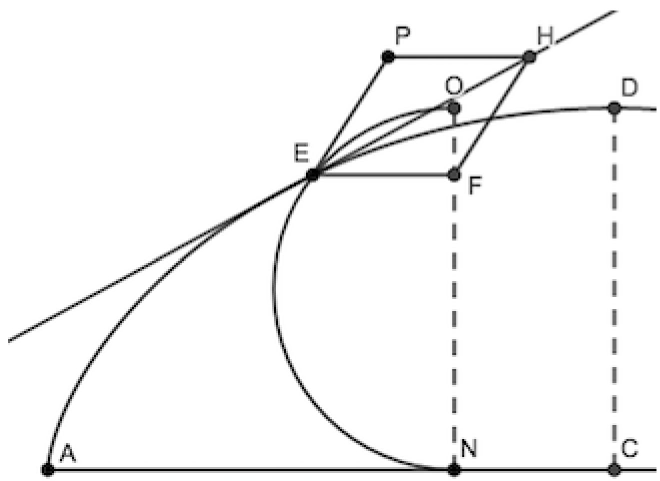

Figure 2: Roberal's Tangent

Proof: Begin with a point E on the cycloid AED and construct the generating circle NEO passing through this point. By definition of a cycloid, the curve is traced out by the point E, and at the same time the vertical diameter of the generating circle NO is moving to the right along the base of the cycloid. Based off these two statements, the point has both a circular and rectilinear motion. Circular motion is defined as the movement of an object as it rotates on the circumference of a circle and rectilinear motion is defined as the movement of an object along a straight line.

According to uniform circular motion, the circular motion of the point is always in the direction of the tangent of the circle, seen here as EP. Simultaneously the rectilinear motion EF is in the direction of the line perpendicular to the vertical diameter of the circle. Being that both these motions are uniform and simultaneous, the ratio of the circumference of the generating circle over the length of the baseline of cycloid is equal to the circular motion over the rectilinear motion.

$$\frac{the\, circumference\, of\, NEO}{the\, length\, of\, AC} = \frac{EP}{EF}$$

Since the generating circle makes one full rotation to form half the cycloid, the circumference of the generating circle must equal the length of the baseline. Therefore the circular motion must equal the rectilinear motion. By the law of parallelogram forces, the diagonal between the motions is the net force of the point on the curve and therefore must be the tangent of the cycloid at that point.

2.2 Fermat

Pierre de Fermat (1607-1665) developed his general method of tangents during the 1630's. Unlike Roberval, his analytical method utilized limits in order to bring two points into coincidence. While Fermat didn't understand the idea of limits the way they are used today, his use of this method demonstrates the imminent arrival of calculus during the 17th century.

Theorem 2.2: The parallel line to the base of the cycloid consisting of point R cuts the generating circle at the circumference and on the diameter. These two points and the highest point on the circle form a right triangle. The hypotenuse of the triangle similar to this consisting of point R and the point intersecting the diameter is the tangent to the cycloid at point R.

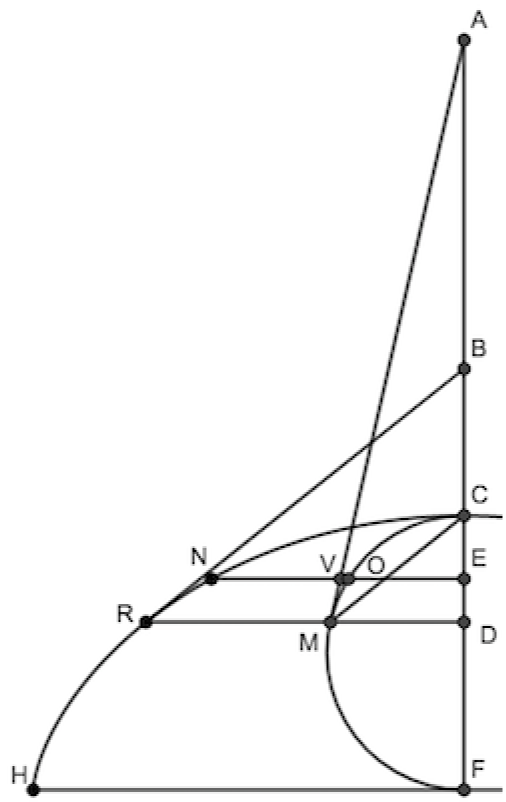

Figure 3: Fermat's Tangent

Proof: Construct the cycloid HRC and generating circle $COMF$. Draw RMD perpendicular to CF, the segment MC, and RB parallel to MC. Once completing the beginning construction of the proof, Fermat's analysis begins by assuming the line RB is the tangent to the cycloid at R. RMD is perpendicular to FC, which cuts the generating circle at point M. Construct MA tangent to the generating circle at M and $EOVN$ parallel to RD.

A brief overview of Fermat's method involves the use of the property of the cy-

cloid, similar triangles, identities, and simple algebra. His logic relies on the assumption that DE approaches zero as a limit, which implies that the points N and R approach coincidence. All this leads to Fermat proving triangles MDC and RDB are similar, which leads to RB being the tangent and parallel to MC.
For further details, the property of the cycloid gives $NO = arc(OC)$. The following substitutions are used for simplicity:

$$DB = a; \qquad DA = b; \qquad MA = d; \qquad MD = r;$$
$$RD = z; \qquad DE = e; \qquad arc(CM) = n; \qquad EB = a - e.$$

Starting with $\frac{a}{a-e}$, multiply by z and move a to the bottom to get $\frac{a}{a-e} = \frac{z}{\frac{za-ze}{a}}$. Then using similar triangles and assuming that N is on RB, $\frac{DB}{EB} = \frac{RD}{NE}$. Since $\frac{DB}{EB} = \frac{a}{a-e}$ from the substitutions above, that means the two functions can be set equal to get $NE = \frac{za-ze}{a}$. Now it is known that $NE = NO + OE$, and it is found using the property of the cycloid: the arc of the generating circle, beginning at the top of the circle, is equal to the line segment from the point on the bottom of the arc to a point on the cycloid parallel to the bottom of the cycloid, to get $NE = arc(OC) + OE = arc(MC) - arc(MO) + OE$. Using substitution again with the equations involving NE gives $arc(MC) - arc(MO) + OE = \frac{za-ze}{a}$.
Using the same method as above and assuming O is on MV, $VE = OE$ and $MV = arc(MO)$. Then similar triangles gives $\frac{MD}{VE} = \frac{DA}{EA}$ and by substitution, $\frac{MD}{VE} = \frac{b}{b-e}$. Multiplying by r and moving b to the bottom gives $\frac{MD}{VE} = \frac{r}{\frac{rb-re}{b}}$. Since it is known that $MD = r$, it follows that $VE = \frac{rb-re}{b}$. It is also known using similar triangles

and substitution that $\frac{DA}{EA} = \frac{MA}{MV} = \frac{b}{e}$, and then multiplying top and bottom by d gives $\frac{DA}{EA} = \frac{d}{\frac{de}{b}}$. Then $MV = \frac{de}{b}$ by setting the two equations together.
Now substituting $arc(MO)$ for MV and OE for VE, $arc(MC) - arc(MO) + OE = arc(MC) - MV + VE$. Using all the equations defined above, it follows that $\frac{za-ze}{a} = n - \frac{de}{b} + \frac{rb-re}{b}$. Applying substitution and arithmetic, the equation simplifies to $\frac{MD+MA}{DA} = \frac{RD}{DB}$. Using the angle bisector theorem and the fact that the chord MC bisects the angle DMA, $\frac{MD}{MA} = \frac{DC}{CA}$. This simplifies down to $\frac{MD+MA}{DA} = \frac{MD}{DC}$. And using this and $\frac{MD+MA}{DA} = \frac{RD}{DB}$ gives $\frac{MD}{DC} = \frac{RD}{DB}$. This proves that triangles MDC and RDB are similar, and therefore the tangent line RB is parallel to MC.

2.3 Descartes

René Descartes (1596-1650) also computed tangents, publishing his general method in 1637. His method was contingent on whether or not he was able to write the equation of the curve. For the cycloidal curves, Descartes would apply a mechanical method relying on the instantaneous center of rotation.

Theorem 2.3: The parallel line to the base of the cycloid consisting of point B cuts the cycloid at the circumference. There is a segment with point B parallel to the segment consisting of the point on the circumference and the lowest point on the cycloid. The line perpendicular to this segment at point B is the tangent to the cycloid.

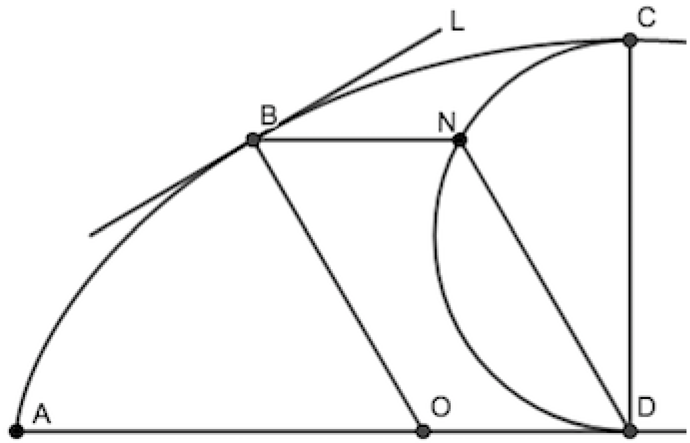

Figure 4: Descartes' Tangent

Proof: Let the cycloid ABC be given along with its generating circle CND. Let B be any point on the cycloid and construct BN parallel to the base AD, cutting CND at N. Construct as well ND, BO parallel to ND, and BL perpendicular to BO. Let the generating circle be a polygon with an infinite number of sides, in reference to Figure 5. The tangent at point B will be the line perpendicular to B and the point in which the generating circle touches the base of the cycloid, in this case D. Since it is known that BN is parallel to AD, ND must be the perpendicular line due to the fact that N will coincide with B. Hence BL is the tangent to the cycloid at point B.

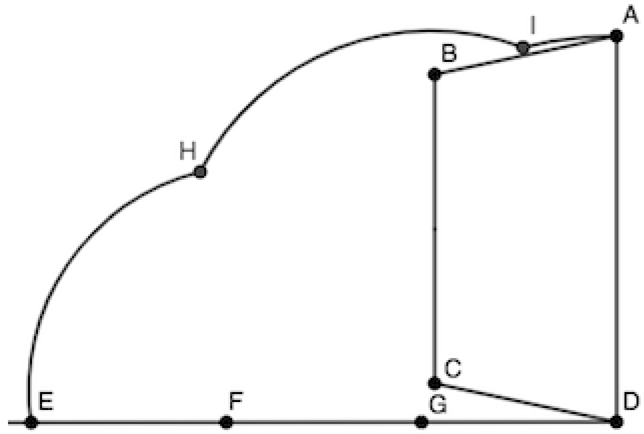

Figure 5: Polygon Rotating on a Straight Line

2.4 Torrcelli

Evangelista Torricelli (1608-1647) was one of the last to construct the tangent to the cycloid in 1644. His method, similar to Roberval, consisted of composition of motion and instantaneous center of rotation. It is unclear on whether this was his original idea, or it came from Roberval since they were in contact through letters around this time period [5]. Either way, Torricelli was unable to apply this method to finding the tangent to the cycloid until his friend, Viviani, showed him how to do it. Torricelli did have additional methods for finding the tangents that are unique.

2.5 Results

The general method for computing the tangent to the cycloid described can be separated into two groups: mechanical and analytic. Mathematicians who provided mechanical methods include Roberval and Torricelli. All three of these methods focused on viewing the point on a curve as a composition of motion. Both Roberval and Torricelli utilized instantaneous center of rotation, but this similarity may be influenced by the circumstance that they were in contact with one another while the methods were being developed. The mathematicians that provided the analytic methods were Fermat and Descartes. They both brought into coincidence two line segments and utilized a method of limits. Though calculus was not fully developed during the period when these methods were produced, it is evident that ideas around the subject were being considered. These proofs for calculating the tangent to cycloid curve helped aid in the development of calculus and related concepts.

3 Quadrature

Developing a general method to determine the area under the curve was also assisted by the study of the cycloid curve. The problem of finding the quadrature of the cycloidal arch has a long history, beginning with Galileo Galilei (1564-1642). In 1599, to determine the area under the curve, Galileo cut out metal in the shape of the cycloid and the generating circle, and then he weighed the two to discover the area of the cycloid is three times the area of the generating circle [3]. While his discovery was correct, he quickly discarded it believing the answer to be only an approximation. Roberval and Torricelli later proved mathematically that the quadrature of the cycloid curve is indeed three times the area of the generating circle.

3.1 Roberval

Gilles de Roberval not only was an early pioneer to the field of tangents, but he also produced one of the earliest quadrature of the cycloid in 1634 which was not published until 1693. His method relied on the companion curve to the cycloid - the sine curve - and Cavalieri's theorem, which states any two figures with the same height and equal width at every height have the same area.

Theorem 3.1: The area under the cycloid is three times the area of the generating circle.

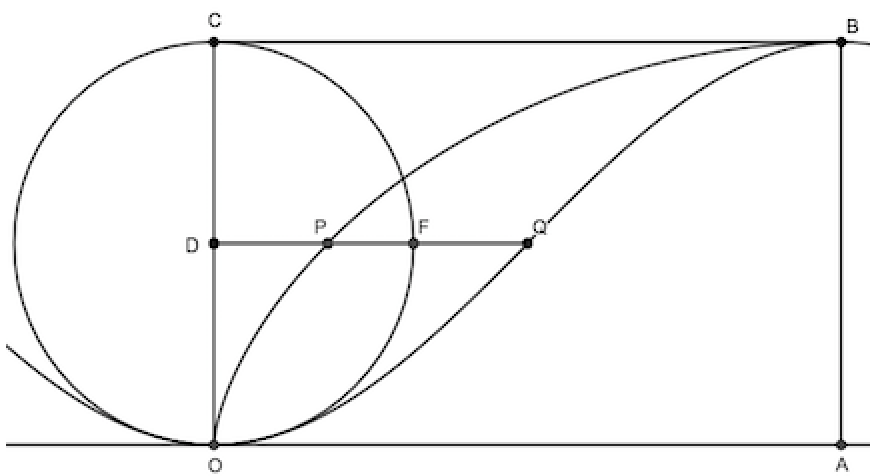

Figure 6: Roberval's Quadrature

Proof: Begin by letting $OABP$ be the area under half the cycloid curve with generating circle of diameter OC and center D. Let P be any point on the cycloid and construct PQ equal to DF. To establish the companion curve, take the locus of points traced by Q as D moves along the diameter of the generating circle. The parametric equations of the companion curve are found to be $x = at$ and $y = a(1 - \cos(t))$. Solving for t in the first equation and plugging the result into the second gives $y = a(1 + \sin(\frac{x}{a} - \frac{\pi}{2}))$. It is found that the curve, OQB, is a sine curve with a being equal to the radius of the generating circle.
It is shown using Cavalieri's theorem that the curve OQB divides the rectangle $OABC$ into two equal parts. It is also known that the base of the rectangle is equal to the semi circumference of the generating circle by construction and the height is equal to the diameter. Therefore the area of the rectangle is twice that of the generating circle, and the area of $OAQB$ is equal to the area of the generating circle. By construction, PQ and DF are equal lengths as DQ moves along the diameter. Hence the widths of the semi-circle OFC and the area between the cycloid OPB and the curve OQB are equal. And since their heights are equal as well, the area of these two segments are the same. So the area under the half arch of the cycloid is one and a half times the area of the generating circle, and the area of the cycloid is three times the area of the generating circle.

3.2 Torricelli

Another successful method of the quadrature of the cycloid curve was done by Evangelista Torricelli, published in 1644. Roberval's method utilized Cavalieri's theorem, which is considered to be an implementation of the method of indivisibles. Torricelli had a similar method using indivisibles.

Theorem 3.2: The area under the cycloid is three times the area of the generating circle.

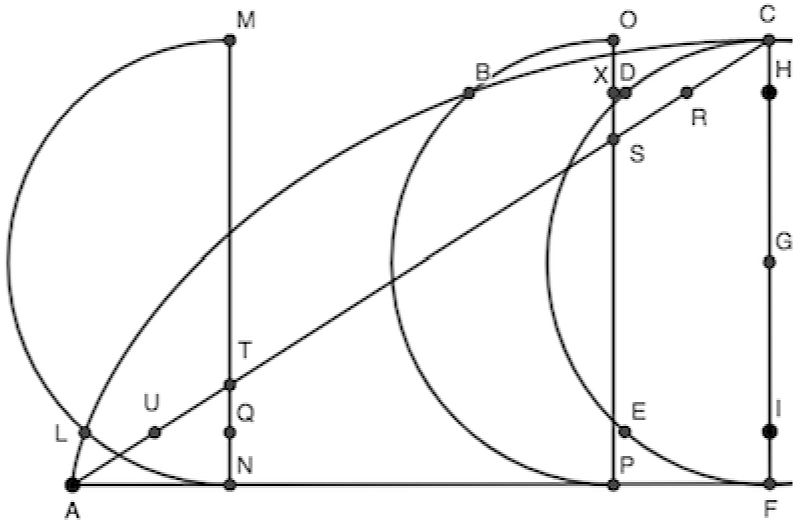

Figure 7: Torricelli's Quadrature

Proof: Let the cycloid curve ABC be given with the generating circle $CDEF$ and base AF. Assume that the area of the cycloid is three times the area of the generating circle, or one and a half times the area of the triangle ACF. To prove this, take two points, H and I, on the diameter of the generating circle equidistant from the center G. Construct the segments HB, IL, and CM parallel to the base FA. Then construct semicircles OBP and MLN of same size as $CDEF$ passing through the points B and L and tangent to the base FA at points P and N.

Since points H and I are equidistant from the center, Euclid's proposition 3.14 states $HD = IE$. And since all semicircles constructed are of equal size to the generating circle, since segments IE and QL are on the same line, and since segments HD and XB are on the same line, segments $HD = IE = XB = QL$. By construction of points H and I and the property of the cycloid, $arc(OB) = arc(LN)$. It is also known by construction that $CH = IF$ and segment HRD is parallel to segment IUL, and since points R and U are on the same line segment AC, $CR = UA$. By definition of the cycloid, $arc(MLN) = AF$. Also using the definition of the cycloid, $arc(LN) = AN$. Using the past two statements, $arc(LM) = NF$. This same argument gives $arc(BP) = AP$ and $arc(BO) = PF$.

It has been previously stated that $arc(BO) = arc(LN)$, $arc(LN) = AN$, and $arc(BO) = PF$, so they are all equal to each other. The same method to show that $CR = UA$ is used to show that $AT = SC$, but this time using the fact that segments OP and MN are parallel. Since $CR = UA$, $UT = SR$. It is known that the two triangles UTQ and RSX have two of the same angles; $\angle UQT = \angle RXS = 90°$ by construction and $\angle TUQ = \angle SRX$ by the alternate angles theorem, therefore the two triangles UTQ and RSX are similar. This means that $UQ = XR$. Using this previous statement that $UQ = XR = LU$, $BR = BX + XR$, and the fact that $BX = LQ$ by construction, then $LU + BR = LQ + BX$. And since $LQ = DH$ and $BX = EI$ by construction, $LU + BR = DH + EI$. This statement will be true for all constructions, as long as H and I are equidistant from the center. Therefore all lines constructed on figure $ALBCA$ will be equal to all lines constructed on the semicircle $CDEF$. This means the area of the figure $ALBCA$ is equal to the area of the semicircle $CDEF$.

Since side AF of triangle ACF is equal to the circumference of semicircle $CDEF$ by construction and side CF is equal to twice the radius of the semicircle, the area of triangle ACF is twice the area of semicircle $CDEF$ by Archimedes property 1. It follows that the area of triangle ACF is equal to the area of the whole circle of diameter CF. Thus the area of the cycloid curve is equal to one and a half times the area of the triangle ACF and three times the area of the generating circle $CDEF$.

3.3 Results

Roberval and Torricelli proved to have similar methods in their quadrature of the cycloid because they both utilized the method of indivisibles. It is seen clearly in Torricelli's method as well as in Roberval's is use of Cavalieri's Theorem, which is thought to be a modern implementation of the method of indivisibles. Cavalieri first used this method of finding areas under curves in 1629 in his submission of his notes on the theory of indivisibles. This new method, and the advances to follow, "exerted an enormous influence upon the subject of finding the areas under curves, hence on the development of the calculus" [6].

4 Rectification

Another sequence of mathematical work pertaining to the cycloid came from Christopher Wren (1632-1723). While he is known mostly for his architectural work, he was a talented mathematician. Newton himself "paid Wren the compliment of ranking him with John Wallis and Christiaan Huygens as a leading geometer of his day" [10]. Wren's proof on the rectification, determining the length of the curve by finding a straight line of equal length, of the cycloid brought him this fame and acknowledgment throughout Europe. Before the 1650's the problem of rectification was thought to be unsolvable, even to Descartes, but Wren correctly found it to be exactly eight times the radius of the generating circle [10]. Unfortunately Wren did not publish his mathematics. However we find information in John Wallis' (1616-1703) *De Cycloide*. In this book Wallis provides a unified published work of Wren's solutions with the cycloid curve, including his rectification of the curve produced in 1658.

4.1 Wren

In order to formulate a proof for the rectification of the cycloid, Wren begins by laying a foundation of problems and lemmas to aid in the proof.

Problem 4.1: Finding the relationship between the cycloidal arc and the internal and external line segments.

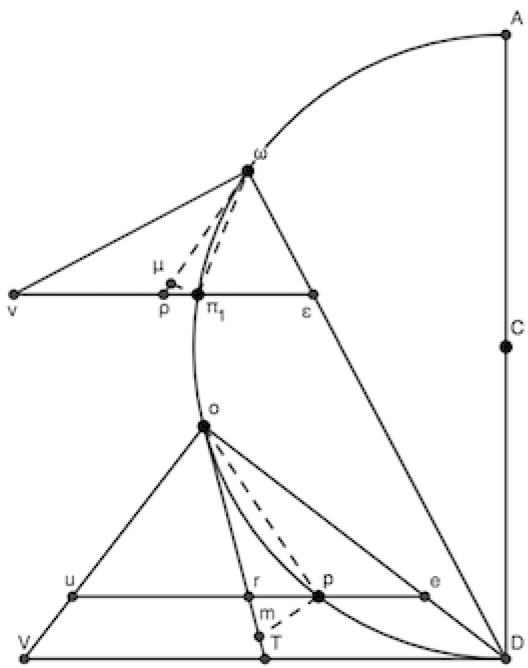

Figure 8: Wren's Problem 1

Proof: Begin with the circle AoD with diameter AD and center C. Construct the tangent VD at point D and draw Vo intersecting V in such a way that Vo and Do make a right angle. Draw the tangent oT at point o and construct triangle oCD. Since segments $oC = CD$ by construction, $\angle CoD = \angle CDo$. And by construction of the tangent line, $\angle ToC$ and $\angle TDC$ are right angles. Because $\angle ToC = \angle ToD + \angle CoD$ and $\angle TDC = \angle TDo + \angle CDo$, $\angle ToD = \angle TDo$. It is also known that $\angle ToV + \angle ToD = 90°$ by construction and $\angle TVo + \angle TDo = 90°$ since the sum of all angles is 180°, therefore $\angle TVo = \angle ToV$. And since they are equal, segments $VT = oT$. Since the segment $VTD = VT + TD = oT + TD$, and since segments $oT + TD > \mathrm{arc}(oPD)$, then $VD > \mathrm{arc}(oPD)$.

Let ω be any point on the circle and construct $D\omega$ and $D\nu$ in such a way that they form a right angle. Let $\omega\rho$ be tangent at ω and construct $urpe$ parallel to VD and $\nu\rho\pi\epsilon$ parallel to VD. Also construct ρo and pm such that they meet at right angles and construct $\rho\omega$ and $\pi\mu$ such that they meet at right angles.

$\angle opu$ and $\angle\omega\pi\nu$ are either acute or obtuse. Let $\angle\omega\pi\nu$ be obtuse. Construct $C\omega$ and CD to form triangle $C\omega D$. Using a similar argument as before, segments $C\omega = CD$ and $\angle C\omega D = \angle CD\omega$. Therefore $\angle\rho\omega D = \angle VD\omega$ since $\rho\omega$ and VD are tangents. And since $\nu\epsilon$ is parallel to VD by construction, $\angle\rho\omega\epsilon = \angle\omega\epsilon\rho$, so $\rho\omega\epsilon$ is isosceles. Again using a similar argument as before, segments $\nu\rho = \rho\omega$. Using Euclid's common notion 5, which states the whole is greater than the part, $\omega\mu < \nu\rho$ and $\nu\rho < \nu\pi$, so $\omega\mu < \nu\pi$.

Construct $\omega\pi$. Since $\omega\pi\nu$ is obtuse based on $\omega\pi\mu$ being a right angle, $\omega\mu < \nu\rho < \nu\pi$. It is known that $\omega\mu + \mu\pi > arc(\omega\pi)$, $\rho\nu + \rho\pi = \nu\pi$, and $\nu\pi > arc(\omega\pi)$, since $\rho\nu > \omega\mu$ and $\rho\pi > \mu\pi$. Let $\angle upo$ be acute and consider triangle omp. Since $\angle opm$ is a right angle by construction, $\angle omp + \angle mop = 90°$. Next consider triangle rpm. $\angle rpm + \angle upo = 90°$ since they add up to opm which is a right angle by construction. Also $\angle upo > \angle ueo$ and $\angle mop < \angle moD$ by Euclid's common notion 5. The complement of $\angle upo$ is less than the complement of $\angle mop$ since $\angle upo > \angle mop$. Then $rm < rp$ since $\angle rpm < \angle rmp$. It is known that $ur = ro$, thus $up > om$, $om > arc(op)$, and then $up > arc(op)$. Since $\angle roe = \angle reo$, $\angle poe < \angle peo$ and thus $pe < po$. And since it is known that $po < arc(po)$, $pe < arc(po)$. This can be shown similarly concerning $\pi\epsilon$. This problem has shown that an internal segment, pe, is less than the arc length, po, and the external segment, up, is greater than the arc length, po.

Problem 4.2: Finding the tangent to the cycloid at a point t.

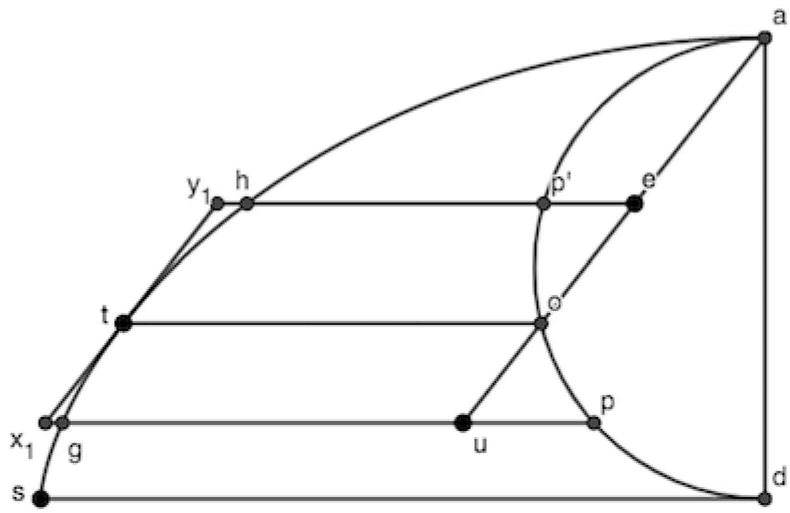

Figure 9: Wren's Tangent

Proof: Construct the semi cycloid sad and its generating circle aod. Draw to parallel to the base and through point o on the generating circle and construct aou and xty parallel to aou.

Say that xy is the tangent at point t. If it is not the tangent, the line must fall inside the curve, either towards the base or towards the vertex. Let the line fall towards the base and let there be some point x inside the curve. It is known that lines parallel to the base that are bounded between the cycloid and generating circle are equal to the arcs of the generating circle cut off from the vertex. Therefore $gp = \mathrm{a}rc(pos)$ and $to = \mathrm{a}rc(oa)$, so $to + \mathrm{a}rc(po) = gp$. Then $xu = to$ by parallel lines, $up > \mathrm{a}rc(op)$ by Problem 1, and $xup > to + \mathrm{a}rc(op)$ by common notion 5.

Since $px > pg$ by the previous statement, x lies outside the curve. But x cannot be both inside and outside the curve, therefore it is not inside the curve.

Let the line fall inside the curve towards the vertex and let there be some point y on ty inside the curve. Draw $yp'e$ parallel to the base and let it cut the cycloid at point h. It is known that $ye = to$ by parallel lines and $hp' + \text{a}rc(p'o) = to$ by definition of a cycloid. And since $p'e < \text{a}rc(p'o)$ by Problem 1, $hp'e < hp' + \text{a}rc(p'o)$ or $hp'e < to$ and $ey > eh$. So y is both inside and outside the curve, therefore y is not inside the curve. No point on xy lies inside the curve, therefore it does not cut the curve at point t and ty is tangent to the cycloid at any given point t.

Problem 4.3: Finding the ratio of subtenses in a given continuous ratio of semi-circles.

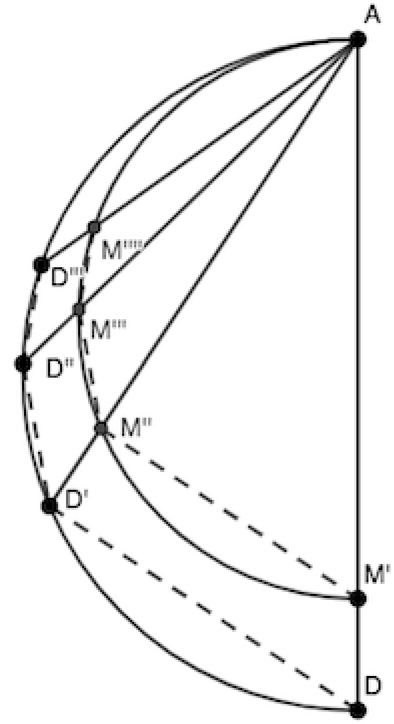

Figure 10: Wren's Subtenses

Proof: Construct the semicircle $AD'D$ and another semicircle $AM"M'$ tangent to $AD'D$ at point A. Construct subtenses, the chords of arcs, such that AD is made equal to AM and join MM and DD everywhere.

Consider triangles $AM''M'$ and $AD'D$. $\angle M''AM' = \angle D'AD$ by common notion 4, which states "things which coincide with one another equal one another," and $\angle AM''M' = \angle AD'D$ by corresponding angles theorem, since M'M" and DD' are parallel by construction. $\angle AM'M'' = \angle ADD'$ by corresponding angles theorem. Therefore triangles $AM'M''$ and ADD' are similar and they have the ratio $\frac{AM'}{AD} = \frac{AM''}{AD'}$. This process can be repeated for all triangles with sides being subtenses, so the ratio AM to AD is continuous throughout the subtenses.

Lemma 4.1: The subtenses in a given ratio can be divided to create equal segments across each subtense.

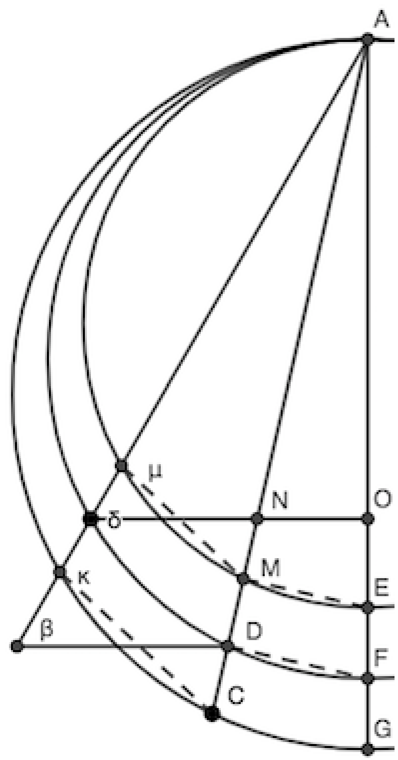

Figure 11: Wren's Lemma

Proof: Construct three circles tangent to each other at point A whose diameters AE, AF, and AG are in continuous proportion. Let line $AMDC$ cut the circles and line $A\mu\delta\kappa$ cut the circles such that $A\delta$ may be made to fit in the middle circle and $AM = A\delta$. Construct δNO perpendicular to the diameter and $D\beta$ parallel to δN and construct κC, μM and ME.

Since $A\mu ME$ is a quadrilateral inscribed in a circle, the external $\angle\delta\mu M$ must be equal to the opposite internal $\angle AEM$. It is knows that $\angle NAO = \angle EAM$ by common notion 4 and $\angle AON = \angle AME$ since $\angle AON = 90°$ by construction and $\angle AME = 90°$ by Thales's theorem, which states the diameter of a circle always subtends a right angle to any point on the circumference of the circle. Therefore by AA similarity theorem triangles AON and AME are similar and $\angle ANO = \angle AEM$. $\angle ANO = \angle\delta NM$ by vertical angle theorem, $\angle\delta\mu M = \angle\delta NM$ by previous statements, and $\angle A\mu M = \angle AN\delta$ because angles on a straight line add up to 180°. $\angle\mu AM = \angle NA\delta$ by common notion 4 and $AM = A\delta$ by construction. Therefore by AAS congruence, triangles $A\delta N$ and $A\mu M$ are equal. Since βD is parallel to δN and κC is parallel to μM, triangles $A\beta D$ and $AC\kappa$ are equal. Therefore $AN = A\mu$, $AM = A\delta$, $AD = A\kappa$, and $AC = A\beta$. $AM = AN + NM$, which implies $NM = AM - AN$, so $NM = A\delta - A\mu$, and $NM = \mu\delta$. This same process shows that $MD = \delta\kappa$ and $DC = \kappa\beta$. Therefore the portions of these subtenses are equal.

Problem 4.4: The diameters of multiple circles tangent to one another are in continuous proportion with one another.

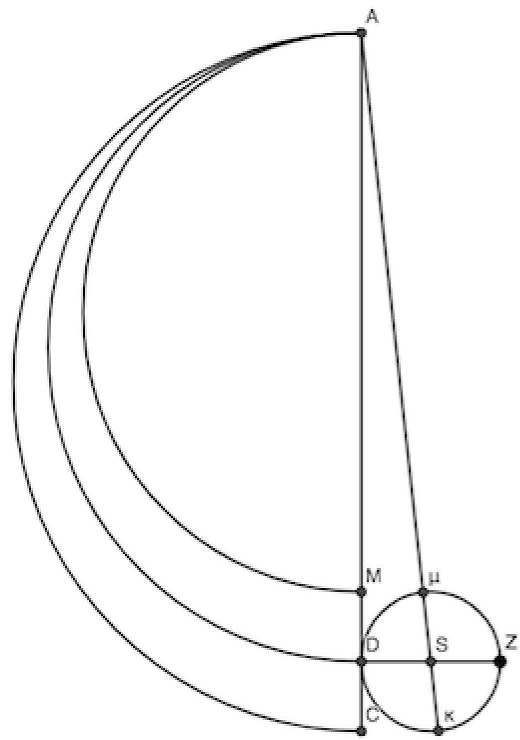

Figure 12: Wren's Problem 4

Proof: Construct three circles all tangent to each other at point A and let AD be the diameter of the middle circle. Construct DZ such that AD and DZ make a right angle and DZ is equal to the difference of the other two circle's diameter, MC. Find the midpoint of DZ, S, and construct circle $D\mu\kappa$ with center S and radius SD. Construct line $A\mu S\kappa$ through S. Let $AM = A\mu$ and $AC = A\kappa$. AD is tangent to the circle $D\mu\kappa$ by construction. By Steiner's theorem, $\frac{AD}{A\kappa} = \frac{A\mu}{AD}$ or $(AD)(AD) = (A\mu)(A\kappa)$.Therefore the square AD is equal to the rectangle $\mu A\kappa$. In substitution, $(AD)(AD) = (AM)(AC)$, so $\frac{AD}{AC} = \frac{AM}{AD}$ and AM, AD, and AC are in continuous proportion.

Lemma 4.2: The sum of the differences of infinitely decreasing magnitudes on a single segment will be equal to the greatest magnitude.

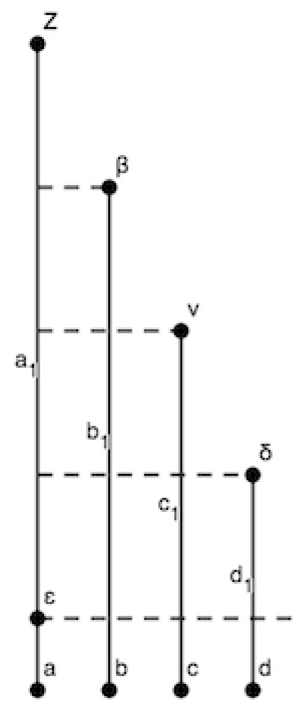

Figure 13: Wren's Magnitudes

Proof: Let the magnitudes aZ, $b\beta$, $c\nu$, $d\delta$, etc. continue infinitely. Let all the differences be greater than aZ such that $aZ = a$, $a\beta = b$, $a\gamma = c$, $a\delta = d$, etc. and $aZ - a\beta = Z\beta$, $a\beta - a\gamma = \beta\gamma$, $a\gamma - a\delta = \gamma\delta$, etc. They are all parts of aZ, therefore the differences can't be greater than aZ. Let the differences be smaller than aZ, and let them be equal to $Z\delta$. But it is known that $d = a\delta$ and $a\epsilon < d$. Since another difference has been added, $Z\epsilon$ is equal to the sum of all the differences. $Z\delta$ must equal $Z\epsilon$ but $Z\delta < Z\epsilon$, so the differences are not smaller than aZ. Therefore

aZ is equal to the sum of all the differences as the magnitudes decrease infinitely.

Lemma 4.3: The difference of two magnitudes is equal to the difference between the sum and twice either one.

Proof: Let a and b be two magnitudes and let $a+b$ be the sum of the magnitudes. Take away twice b from the sum to get $(a + b) - 2b$. Simplifying this gives $a - b$, which is the difference of the magnitudes. Repeat this for twice a to get $(a + b) - 2a = -(a - b)$.

Problem 4.5: The difference of the sum of all the sides of an inscribed polygon and the sum of the sides of a circumscribed polygon is equal to a given magnitude.

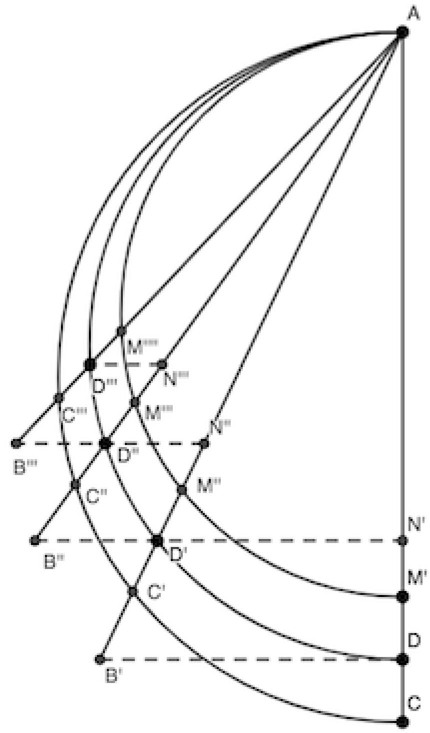

Figure 14: Wren's Serrated Polygons

Proof: Construct the semicircle ADD with diameter AD, and let the given magnitude be X. Also construct semicircles $AM'M''$ and ACC such that AD is the average of AM' and AC and AM' is the smallest diameter. Also let the continuous proportions AM', AD, and AC be constructed such that the differences of extremes, $M'C$, is equal to the given magnitude X. Construct the subtenses AD in such a way that they be placed everywhere such that the ratio of AD to AM to AC is continued infinitely. Once the subtenses have been constructed, draw the parallel segments NDB through each D to the next subtense, such that N precedes the subtense of D and B follows it and NDB is perpendicular to the diameter AD. Based on this construction, a serrated polygon has been inserted into the semicircle ADD and another has been put around the semicircle. A serrated polygon is "a figure which alternates in turn from sides parallel to themselves to sides crosswise with respect to themselves. However, only non-parallel sides may be plainly called sides" (Wallis). Construct the semicircles with diameters AM' and AC tangent at A, and let them intersect the subtenses everywhere at M and C respectively.

Based on construction, subtenses AD are places everywhere in the given ratio AD to AM'. From Problem 3, it can be shown that the preceding subtense AM' is equal to the following one AD', continued infinitely. Then by Lemma 1, it is found that the preceding subtense $M'N'$ is equal to the following $D'M''$ everywhere infinitely. Using this statement it is clear to see that the difference of DM' and $M'N'$ on the same subtense is equal to the difference of DM' and $D'M''$ also on the same subtense everywhere. Using Lemma 2, it is clear to see that the subtenses DM are decreasing infinitely, so the sum of their differences are equal to the maximum DM. It has been said before that the differences of lines DM and MN are equal to the differences of DM' and $D'M''$, and so on, so the sum of all the differences of lines $DM - MN$ is also equal to the maximum DM. By Lemma 3, $DM' - M'N' = 2DM' - DN'$. And the sum of all lines DM is the same as the sum of all differences AD, so they are both equal to the maximum AD. Using Lemma 3 again, the sum of the difference of DM and MN is equal to twice the sum of DM minus the sum of DN, which thus is equal to max DM from a previous statement. And since the sides of the polygon are DN, the sides of the inserted polygon fall short of double the diameter AD by DM'.

Using the same logic since AM', AD, and AC are in continuous proportion, the preceding subtense CD is equal to the following subtense $B'C'$ infinitely. And using the same argument as above, it is shown that twice the sum of lines BC minus the sum of sides BD is equal to the maximum $B'C'$. So all the sides of the circumscribed polygon (BD) falls short of double the diameter AC by $B'C'$. And by Lemma 1 it is known that $AB' = AC$ and from above $B'C' = CD$, therefore all the sides of the circumscribed polygon surpass double the diameter by an excess of DC.

Since the inscribed polygon falls short of double the diameter AD by DM' and the circumscribed polygon surpassed double the diameter by DC, the difference of the two is equal to CM'. This is put equal to the magnitude X, and the problem has been proven.

Lemma 4.4: Relates the length of arcs on the cycloid to a segment of the tangent of the cycloid.

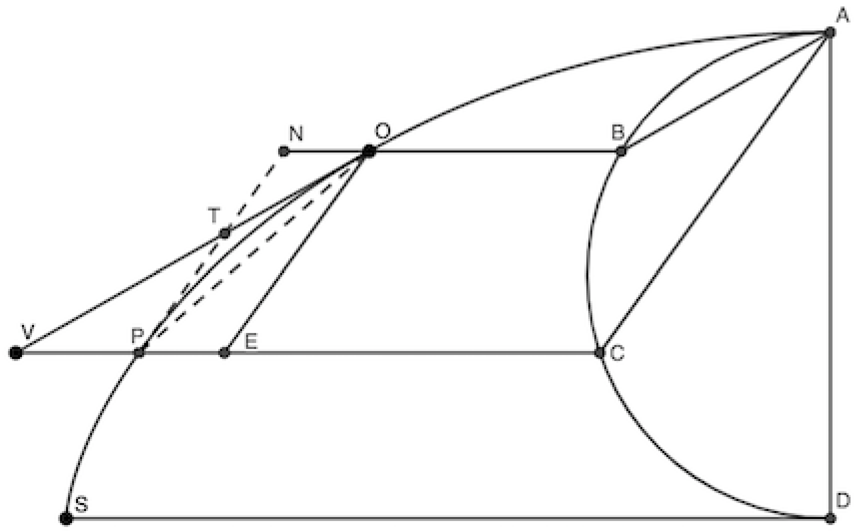

Figure 15: Wren's Lemma

Proof: Let O be a point on the cycloid $SPOAD$ and construct its generating circle. Construct the tangent OV drawn towards the base, and also construct NOB and VPC parallel to the base and let the lines cut the generator at B and C. Construct the segments AB and AC, and construct OE parallel to AC and construct PTN parallel to OE and the segment PD.

$\angle PCA$ is obtuse by construction, so $\angle PEO$ is obtuse since AC and OE are parallel. $EO < PO$ by definition of triangles since $\angle PEO$ is the largest angle. $EO <$ a$rc(PO)$ by definition of chord length since $PO <$ a$rc(PO)$. PN is tangent to the cycloid at P since it is parallel to AC. And since it is parallel and $\angle PCA$ is obtuse, $\angle VPT$ must be obtuse. $TP < TV$ by definition of an obtuse triangle. It is known that $TP + OT < TV + OT$ or $OTV > OT + TP$. Also, $OTV >$ a$rc(OP)$ since $OT + TP >$ a$rc(OP)$ by definition of arc length. So therefore $OV >$ a$rc(OP)$

and $OE < \text{a}rc(OP)$.

Lemma 4.4: The arc length of the cycloid is four times the diameter of the generating circle.

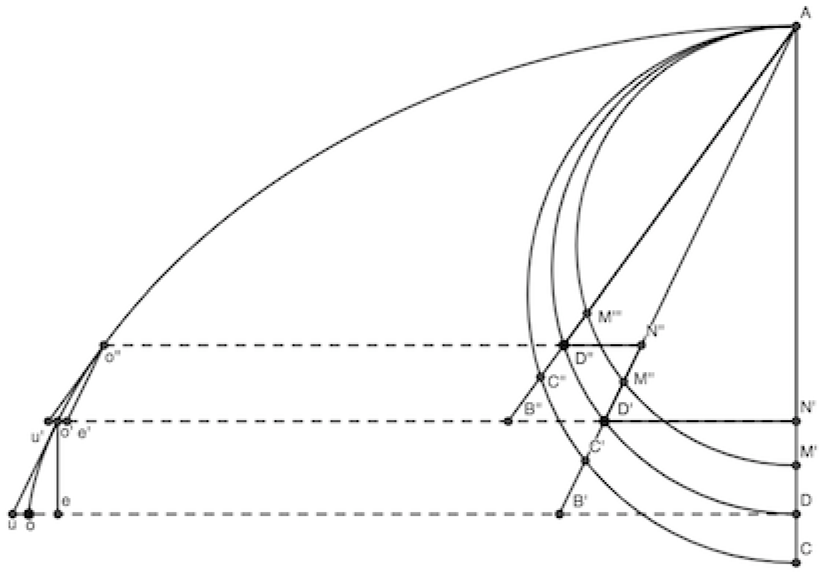

Figure 16: Wren's Rectification

Proof: Construct the semi cycloid ooA and semi generating circle $AD'D$ with base oD and diameter AD. Let twice AD be called X. Construct two more semicircles with $AD'D$ as the middle circle and the ratio of the diameters being constant. Construct a serrated polygon inside the generating circle and let another be placed around it so that the difference of the sum of the sides of the two is equal to X. Let the parallel sides of the polygon be extended so that they cut the curve at oo. Construct the tangent ou and let the infinite serrated polygon be circumscribed around the cycloid. Construct the lines oe such that they are parallel to AD. Let an infinite serrated polygon be inserted into the cycloid.

Since the lines ou are tangent to the cycloid, they are parallel and equal to the lines DB each to the other, since they are both constrained by parallel lines. Therefore the sides of the polygon circumscribed around the cycloid must be equal to the sides of the polygon which are circumscribed around the generating circle. And since the lines oe are parallel and equal to the lines ND, the sides of the inserted polygon into the cycloid must be the same as those of the one inserted into the generating circle. Therefore the sides of the circumscribed polygon around the cycloid exceed double the diameter by an excess of DC, based on Problem 5.

Say the curve is smaller than the sides of the circumscribed polygon. Then by Problem 5 the sides of the polygon inserted into the cycloid are less than twice the diameter by a defect of DM'. Say the curve is greater than the sides of the polygon inserted into the cycloid. Then again by Problem 5 the sides of the circumscribed polygon around the cycloid exceed double the diameter by an excess of DC. Therefore the curve of the semi-cycloid is neither more nor less than twice the diameter. Thus the curve of the primary cycloid is four times the diameter of the generating circle.

Corollary 4.1: The arc length of any portion of the curve cut back to the vertex is two times the subtense drawn from the section of the base of the portion and of the generator.

The theorem proved the arc length of the cycloid to be four times the diameter of the generating circle. Half the arc length is two times the diameter of the generating circle. Using the method from the theorem and the same figure, it is proven that the arc length of any portion of the curve cut back to the vertex (such as oo'A) is two times the subtense AD'.

4.2 Results

Wren's main approach to solving the length of the cycloid curve utilized infinite series. While he used the simplest of infinite processes, his method applies ideas from calculus that are recognizable today. At the time there was no definitive process used regarding infinite series, but the cycloid curve was beneficial in developing the thought process for it. This deomnstrates once again how the idea of calculus was in most mathematicians' minds during the 17th century.

5 Brachistochrone Problem

In June 1696 Johann Bernoulli (1667-1748) posed the brachistochrone problem in *Acta Eruditorum*. With an introduction directed to the leading mathematicians of his time, he poses the following: "given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time" [8]. The mathematicians who solved this problem, including Johann Bernoulli, Jakob Bernoulli, Leibniz, and Newton, discovered the cycloid curve to be the curve of quickest descent. Prior to their discoveries, Galileo worked to solve the quickest descent curve in his 1638 paper *Dialogues Concerning Two New Sciences* [8]. While Galileo was correct in assuming the curve would be some arc of infinite line segments, he was incorrect in deducing the arc to be that of a circle. Despite his incorrect solution, it was still significant in the development of the correct solution. The following proofs are important because they proved to be fundamental to the birth and development of calculus of variations.

5.1 Johann Bernoulli

Johann Bernoulli, a Swiss mathematician, was the first of the 17th century mathematicians to solve the problem in 1696. His proof relies on considering the path of a light ray through a medium of arbitrarily varying densities, such that it becomes less dense from top to bottom. So as light enters the medium from above, the speed becomes increasingly quicker as it moves down. Along with these conditions, Johann Bernoulli's proof relies on Fermat's minimum time principle and Snell's law.

Theorem 5.1: The solution to the Brachistochrone Problem is the cycloid curve.

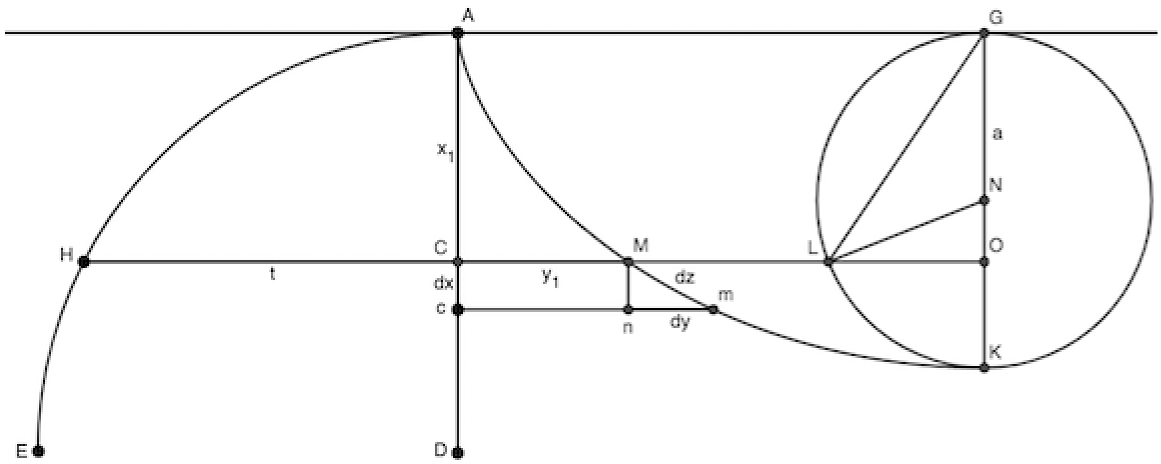

Figure 17: Johann Bernoulli's Brachistochrone Problem

Proof: Define AGD to be the medium bounded by the horizontal line AG where A is the luminous point, or point of light. Let the curve AHE be given with vertical axis AD, and let the ordinate HC determine the density of the medium at height AC. Construct the curve AMK and the circle GLK. Set $x = AC$, $y = CM$, and $t = CH$, and for the infinitesimals let Cc, mn, and Mm be equal to dx, dy, and dz respectively.

Let angle nMm, or θ_r, be the angle of refraction at M such that $\theta_r = \frac{nm}{Mm} = \frac{dy}{dz}$. According to Snell's law, the ratio of the sine of the angle formed by the incident ray over the velocity of light in that ray is equal to the sine of the angle formed by the refracting ray over the velocity of light in that ray. Using this law, for the path we want to find, this quantity has to be proportional to the velocity t such that $\frac{dy}{dz} = \frac{1}{a}t$ for some positive constant a. Using this formula and the Pythagorean

Theorem for $dz^2 = dx^2 + dy^2$, the following holds:

$$dy^2 = \frac{t^2}{a^2}dz^2 = \frac{t^2}{a^2}dx^2 + \frac{t^2}{a^2}dy^2 = \frac{t^2}{a^2 - t^2}dx^2$$

$$\frac{dy}{dx} = \frac{t}{\sqrt{a^2 - t^2}} \tag{1}$$

Galileo's law of falling bodies states that in a vacuum the velocity t is proportional to the square root of the falling height. Therefore set $t = \sqrt{ax}$, where a is just a constant and x is the height AC, and substitutes it into equation (1) to get:

$$dy = \sqrt{\frac{x}{a - x}}dx \tag{2}$$

Johann Bernoulli recognized this to be the differential equation of the cycloid curve, so he concluded that the brachistochrone is the ordinary cycloid. This claim is dependent on being able to integrate the differential equation, which is done by rewriting equation (2) as the following:

$$dy = \frac{1}{2}\frac{adx}{\sqrt{ax - x^2}} - \frac{1}{2}\frac{a - 2x}{\sqrt{ax - x^2}}dx \tag{3}$$

The second term on the right hand side can be integrated as followed:

$$\int \frac{a - 2x}{2\sqrt{ax - x^2}}dx = \sqrt{ax - x^2} + C \tag{4}$$

Let L be the point where the segment HC and the circle GLK intersect. Construct LG and LN where LN is the radius of the circumference GLK with diameter $GK = a$ and $GO = x$. Consider triangle LNO, where it follows that $LN = \frac{a}{2}$

and $NO = a - \frac{a}{2}$. Using the Pythagorean Theorem for triangle LNO, the following equations are found:

$$LO^2 = (\frac{a}{2})^2 - (x - \frac{a}{2})^2$$

$$LO = \sqrt{ax - x^2} \tag{5}$$

Next Johann Bernoulli claims that the first term on the right-hand side of the differential equation is the differential a$rc(GL)$.

$$\int \frac{a}{2\sqrt{ax - x^2}} dx = \int \sqrt{\left[d(\sqrt{ax - x^2})\right]^2 + dx^2} = \int \sqrt{dLO^2 + dOG^2}$$

$$\int \frac{a}{2\sqrt{ax - x^2}} dx = \text{a}rc(GL) \tag{6}$$

Utilizing the derived equations, it is inferred that

$$CM = y = \int dy = \text{a}rc(GL) - LO$$

And since it is know

$$\begin{aligned} MO = CO - CM &= CO - \text{a}rc(GL) + LO \\ &= semicircle\, GLK - \text{a}rc(GL) + LO \\ &= \text{a}rc(LK) + LO \end{aligned}$$

and $MO = ML + LO$, it can be concluded that a$rc(LK) = ML$. Johann Bernoulli concludes his prove by saying, "by taking into account the definition of the cycloid, the equation readily yields that the curve AMK that solves the differential equation is a cycloid" [2].

5.2 Jakob Bernoulli

Jakob Bernoulli (1655-1705), Swiss mathematician and brother of Johann Bernoulli, accepted the challenge of solving the brachistochrone problem in 1696. In contrast to his brother's solution, Jakob Bernoulli had a more methodical and general approach that later became the basis of calculus of variations. While he doesn't use calculus specifically, both he and his brother utilize the differential equation to solve the problem. His influential proof relies on "the use of similar triangles, some hand-waving regarding infinitesimals, and the concept of stationary points of functions" [1]. A stationary point of a function, also known as the local extrema, is a point where the rate of change is zero for that function. And since the brachistochrone minimizes descent time, "the rate of change of descent time must be zero with respect to infinitesimal variation of the brachistochrone path" [1].

Theorem 5.2: The solution to the Brachistochrone Problem is the cycloid curve.

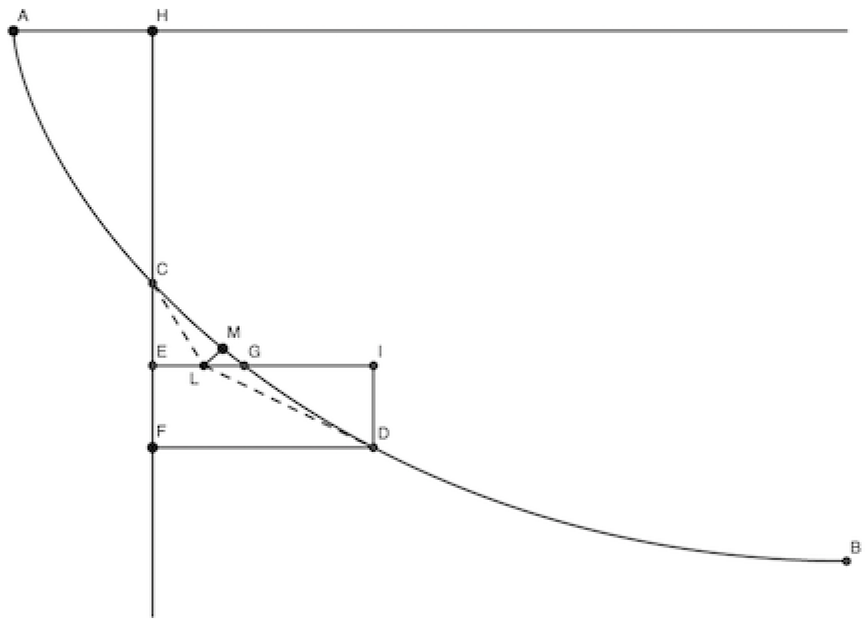

Figure 18: Jakob Bernoulli's Analytic Method

Proof: Let AB be the least descent curve in Figure 18 with two points, C and D, on the curve. Jakob Bernoulli claims that a$rc(CD)$, out of all the arcs, is the arc where a falling body travels through the fastest. If, say, the body traveled quicker on a sub a$rc(CED)$, then $ACEDB$, instead of $ACDB$, must be the curve of quickest descent. Let C and D be infinitely close on the curve $ACDB$ and point L be on the segment EI, where E is the midpoint between C and F, in such a way that GL is the differential of GE. Also let lines AH, EI, and FD be parallel and HF be orthogonal to them.

Since CGD is the minimizing arc, by construction, through C and D, then

$$t_{CD} + t_{LD} = t_{CG} + t_{GD}$$

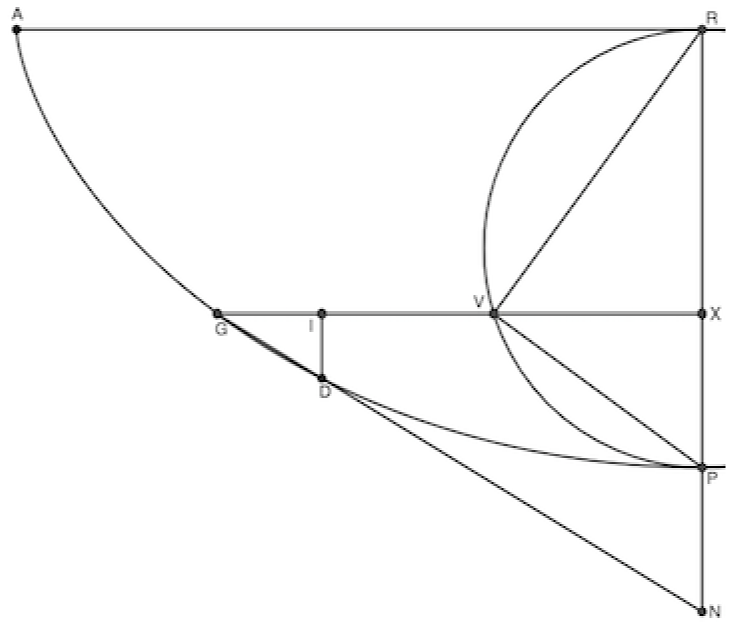

Figure 19: Jakob Bernoulli's Geometric Method

where t_{CD} is the time descent from C to D. Moving the equation around gives

$$t_{CG} - t_{CL} = t_{LD} - t_{GD} \tag{7}$$

If CG is considered an inclined plane, the law of the inclined plane, which says the velocity of a falling object will be equal over any inclined plane as long as the heights are equal, gives

$$\frac{CE}{CG} = \frac{t_{CE}}{t_{CG}}$$
$$\frac{CE}{CL} = \frac{t_{CE}}{t_{CL}}$$

Euclid's proposition V.24 lets Jakob Bernoulli conclude

$$\frac{CE}{CG-CL}=\frac{t_{CE}}{t_{CG}-t_{CL}} \tag{8}$$

Let LM be perpendicular to CG where M intersects CG. Since GL is an infinitesimal of higher order with respect to EG, it can be assumed that $CG-CL=MG$. Consider triangles MLG and CEG. They are similar triangles because they both have a right angle by construction and they share a similar angle: angle CGE and angle LGM. This conclusion yields

$$\frac{EG}{CG}=\frac{MG}{GL}$$

Multiply both sides of the equation by $\frac{t_{CE}}{t_{CG}-t_{CL}}$ to get:

$$\frac{EG\bullet t_{EC}}{CG(t_{CG}-t_{CL})}=\frac{CE}{GL} \tag{9}$$

This same process with the segment NG being perpendicular to LD, where $LN=LD$–GD and triangles GID and LNG are similar triangles, gives

$$\frac{GI\bullet t_{EF}}{GD(t_{LD}-t_{GD})}=\frac{EF}{GL} \tag{10}$$

Comparing the equations (9) and (10), while taking into consideration equation (7) and that $CE=EF$, gives the following:

$$\frac{EG\bullet t_{EC}}{GI\bullet t_{EF}}=\frac{CG}{GD}$$

According to the gravity law, which states that $T = \sqrt{\frac{2}{g}} \frac{l}{\sqrt{h}}$, it is concluded that

$$\frac{CG}{GD} = \frac{\frac{EG}{\sqrt{HC}}}{\frac{GI}{\sqrt{HE}}} \tag{11}$$

Therefore each segment of the minimizing curve is directly proportional to the abscissa and inversely proportional to the square root of the ordinate. This property belongs to and characterizes the isochronous curve of Huygens, which means the curve of quickest descent is the cycloid curve.

Along with this analytic proof, Jakob Bernoulli provides a geometric solution. Consider Figure 19 where AGP is the cycloid with generating circle RVP. It is apparent that the two triangles GDI and GNX are similar because they share two angles, therefore $\frac{GD}{GI} = \frac{GN}{GX}$. Then by definition of a tangent to the cycloid and since G and V are on the same line segment, GN is parallel to VP, which means they have three angles in common. Therefore triangles GNX and VPX are similar and $\frac{GN}{GX} = \frac{VP}{VX}$. Next, it is known that triangles VPX and VRX are similar, which means $\frac{VP}{VX} = \frac{VR}{RX}$. According to the geometric mean leg theorem, $VR = \sqrt{(RP)(RX)}$. Using substitution and the fact that $HE = RX$ gives:

$$\frac{VR}{RX} = \frac{\sqrt{RP}}{\sqrt{HE}}$$

Putting all these equations together gives the following:

$$\frac{GD}{GI} = \frac{GN}{GX} = \frac{VP}{VX} = \frac{VR}{RX} = \frac{\sqrt{RP}}{\sqrt{HE}} \tag{12}$$

A similar method will show

$$\frac{EG}{CG} = \frac{CS}{CM} = \frac{QS}{QP} = \frac{RS}{RQ} = \frac{\sqrt{RS}}{\sqrt{RP}} = \frac{\sqrt{HC}}{\sqrt{RP}} \tag{13}$$

Putting equations (12) and (13) together gives:

$$\frac{GD}{CD} = \frac{GI\sqrt{RP}\sqrt{HC}}{GI\sqrt{HE}\sqrt{RP}} = \frac{GI\sqrt{HC}}{EG\sqrt{HE}} \tag{14}$$

This gives the same formula as the analytic approach.

Finishing up the analytic approach, Jakob Bernoulli sets the following values, $CG = ds = \sqrt{dx^2 + dy^2}$, $HE = x$, $CE = dx$, and $EG = dy$, and plugs it into the equation (11) to get

$$ds = \frac{k}{\sqrt{x}} dy$$

That is

$$\frac{dy}{dx} = \sqrt{\frac{x}{k^2 - x}} \tag{15}$$

Therefore the solution is the cycloid.

5.3 Leibniz

Gottfried Wilhelm Leibniz (1646-1716), a prominent German mathematician, sent a letter to Johann Bernoulli containing his solution to the brachistochrone problem on 16 June 1696. Before beginning his proof, he claims that the curve they are looking for is one where each line segment of the curve is directly proportional to the latitude and inversely proportional to the square root of the altitude: $ds =$

$\frac{k}{\sqrt{x}}dy$. And since it is known that $ds^2 = dx^2 + dy^2$, this equation can be manipulated to get

$$\frac{dy}{dx} = \sqrt{\frac{x}{k^2 - x}}$$

This is the same method Jakob Bernoulli used to prove the equation he found was the cycloid.

Theorem 5.3: The solution to the Brachistochrone Problem is the cycloid curve.

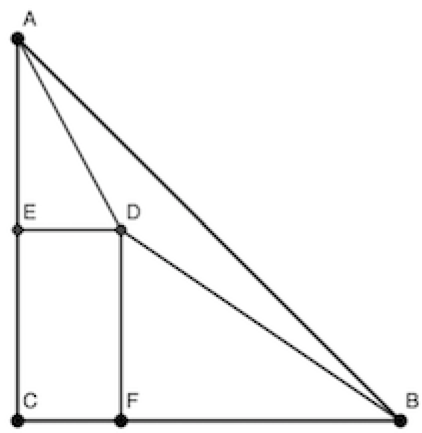

Figure 20: Leibniz's Triangle

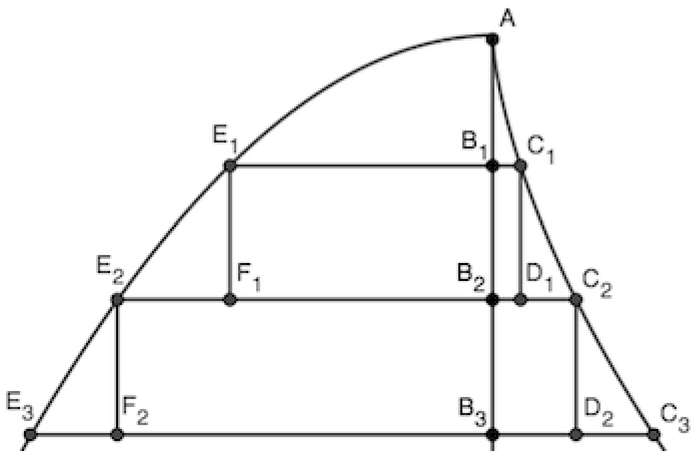

Figure 21: Leibniz's Parabola

Proof: Construct the triangle ABC. Bisects the height AC at E and find the point D on the horizontal line through E and parallel to CB in such a way that the quickest descent curve is the path ADB.

According to Galileo's law of falling bodies

$$t_{AE} = \sqrt{\frac{AE}{AC}} t_{AC} \qquad t_{EC} = \left(1 - \sqrt{\frac{AE}{AC}}\right) t_{AC}$$

Then by the law of the inclined plane,

$$t_{AD} = \frac{AD}{AE} t_{AE} = \frac{AD}{AE} \sqrt{\frac{AE}{AC}} t_{AC}$$

and

$$t_{DB} = \frac{DB}{EC} t_{EC} = \frac{DB}{EC} \left(1 - \sqrt{\frac{AE}{AC}}\right) t_{AC}$$

The time it takes to go from A to D and from D to B is

$$t_{ADB} = \left[\frac{AD}{AE}\sqrt{\frac{AE}{AC}} + \frac{DB}{EC}\left(1 - \sqrt{\frac{AE}{AC}}\right)\right] t_{AC}$$

Since D is the only varying point, on the quantities AD and DB vary when D varies.

It is also known that $DB^2 = EC^2 + (CB - ED)^2$ and $AD^2 = AE^2 + ED^2$ from the Pythagorean Theorem. Rewriting the previous equation of while substituting in new values gives

$$t_{ADB} = \left[\frac{\sqrt{AE^2 + ED^2}}{AE}\sqrt{\frac{AE}{AC}} + \frac{\sqrt{EC^2 + (CB - ED)^2}}{EC}\left(1 - \sqrt{\frac{AE}{AC}}\right)\right] t_{AC}$$

Differentiate the equation and set it equal to zero to get the following:

$$\frac{ED}{AD^2} t_{AD} = \frac{FB}{DB^2} t_{DB} \tag{16}$$

Given this equation, Leibniz constructs Figure 21. Construct parabola AE with vertex A and axis AB, such that a body falls vertically from A to B in the time BE. Let AC be the brachistochrone and B_1, B_2, and B_3 be equidistance apart. If AC is the brachistochrone and B_1, B_2, and B_3 are equidistant, then

$$t_{C_1C_2}\frac{D_1C_2}{(C_1C_2)^2} = t_{C_2C_3}\frac{D_2C_3}{(C_2C_3)^2}$$

using equation (16). According to Galileo's law,

$$t_{C_1C_2} = \frac{C_1C_2}{C_1D_1} t_{C_1D_1} \qquad\qquad t_{C_2C_3} = \frac{C_2C_3}{C_2D_2} t_{C_2D_2}$$

Plug these two equations into the previous one to get

$$\frac{D_1C_2}{C_1C_2} t_{C_1D_1} = \frac{D_2C_3}{C_2C_3} t_{C_2D_2}$$

since $C_1D_1 = C_2D_2$. Manipulating the equation more gives

$$\frac{D_1C_2}{D_2C_3} = \frac{C_1C_2}{C_2C_3} \bullet \frac{t_{C_2D_2}}{t_{C_1D_1}}$$

It is known that $t_{AB_1} = B_1E_1$, $t_{AB_2} = B_2E_2$, $t_{AB_1} + t_{B_1B_2} = B_2E_2$, $t_{AB_1} + t_{B_1B_2} = B_1E_1 + F_1E_2$, and $t_{B_1B_2} = F_1E_2$. Using this, it can be found that

$$\frac{D_1C_2}{D_2C_3} = \frac{C_1C_2}{C_2C_3} \bullet \frac{F_2E_3}{F_1E_2}$$

Since $D_1C_2 \propto dy$, $F_1E_2 = t_{B_1B_2} \propto \sqrt{x}$, and $C_1C_2 \propto ds$, the initial equation $ds = \frac{k}{\sqrt{x}} dy$ is proven and the curve of least descent is the cycloid.

5.4 Newton

Sir Isaac Newton (1643-1727), an English mathematician published his solution anonymously twice, once in the January 1697 issue of the Philosophical Transactions and once in the May 1697 issue of Act Eruditorum. Unfortunately, "in Newton's paper appears no reason as to why the solution should be a cycloid and there is anywhere no record of the method followed by Newton to face Bernoulli's challenge to be found" [2]. While there is little information to be found on Newton's method, it can be inferred that he reasoned in geometric terms and it is suggested that it is similar to Johann Bernoulli's method.

5.5 Results

One similarity found between the methods of each mathematician was their use of proportions. Both Leibniz and Jakob Bernoulli found that the element of line is directly proportional to the element of latitude and inversely proportional to the square root of the altitude. This proportion gives the differential equation of the cycloid curve. These three also had a similar method in which they would find properties unique to the cycloid in order to determine the curve of quickest descent is the cycloid. And when comparing the methods used to solve the brachistochrone problem, it's important to focus on the relations to calculus. Both Johann and Jakob Bernoulli had different methods, but they ended the same way in which they found the differential equation of the cycloid curve and declared the problem to be solved. Leibniz too had a similar method where he found the differential equation and knew the solution to be the upwards-facing cycloid. It's not surprising that the methods involved differential equations, an integral part of calculus, because by this time calculus was known. In fact, their ability to recognize the curve by the differential equation demonstrates the spread of the subject, along with adding to that spread. Leibniz claimed that it was calculus that gave him the curve he sought and Jakob Bernoulli's method is said to be the basis of the calculus of variations.

6 Conclusion

The cycloid curve was integral in the development of 17th century mathematics. New methods of tangents, areas, and arc lengths to the curve were able to be discovered because these methods could be found without the use of an equation . And because the curve possessed such unique qualities, it was useful in adapting a general method that could be applied to any curve. Among the many great qualities of the cycloid curve, one that stands out is its support in the development of calculus. During the early to mid-17th century calculus was in the air and in the minds of many distinguished mathematicians. Their work with the cycloid curve helped to bring it out into the world for others to see, improve upon their methods, and develop entirely new methods. Most mathematicians discussed were in contact with one another and were able to view each other's methods. All the work they did became an introduction to what is used in calculus today. Their original solutions required extensive thought and insight, but with the invention of calculus they have become simple exercises.

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